

Non-unique conical and non-conical tangents to rectifiable stationary varifolds in \mathbb{R}^4

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December 30, 2012 Received: date / Accepted: date

Abstract We construct a stationary 2-varifold in \mathbb{R}^4 with non-conical, and hence non-unique, tangent varifold at a point. This answers a question of L. Simon (Lectures on geometric measure theory, 1983, p. 243) and a related question of W. K. Allard (On the first variation of a varifold, Ann. of Math., 1972, p. 460). The varifold can be rectifiable.

There is also a (rectifiable) stationary 2-varifold in \mathbb{R}^4 that has more than one conical tangent varifold at a point.

Keywords stationary varifold · varifold tangent · tangent cone · non-unique · non-conical · minimal surface · regularity

Mathematics Subject Classification (2010) 28A75, 49Q20, 35B65

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Research supported by grants P201/12/0290 of GA ČR, IAA100190903 of GA AV and RVO: 67985840.

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1 Introduction

The purpose of this paper is to answer questions of L. Simon [6, p. 243] and W. K. Allard [1, p. 460]. The book [6] and the paper [1] are standard sources cited when varifolds and related regularity results are of concern. Varifolds are generalized (non-oriented) surfaces and admit compactness properties suitable to approach the problem of existence of surfaces with minimal area. For questions about the regularity of the minimizer, the exploration of the tangents is important.

On p. 243, L. Simon recalls the definition of tangent varifolds. He proves that if C is a tangent varifold (and if some natural conditions are satisfied), then μ_C is conical, where μ_C denotes the measure in \mathbb{R}^n associated with C by the direction-forgetting projection $G_m(\mathbb{R}^4) \rightarrow \mathbb{R}^n$. He says that it seems to be an open question whether C itself has to be conical.

Likewise, W. K. Allard [1, p. 459–460] states that all $C \in \text{VarTan}_a V$ are conical (under some conditions on density) and then he says he knows of no varifold (stationary, ...) such that $\text{VarTan}_a V$ has more than one element.

The result that we prove in this paper is the following (see Theorem 5.1 and Theorem 5.2).

Theorem 1.1 *There exists a stationary rectifiable 2-varifold in \mathbb{R}^4 that has a non-conical (hence non-unique) tangent at a point. There exists a stationary rectifiable 2-varifold in \mathbb{R}^4 that has a conical but non-unique tangent at a point. (The varifolds have a positive and finite k -dimensional density at the point.)*

Note that there is no such varifold V with non-conical tangent and $\theta^2(\mu_V, \cdot)$ bounded away from zero on $\text{spt } \mu_V$, as the following results imply.

Lemma 1.1 *Let V be a stationary m -varifold on an open set $\Omega \subset \mathbb{R}^n$, $x_0 \in \Omega$, $C \in \text{VarTan}_{x_0} V$ and $C \neq 0$.¹*

If $\theta^m(C, x) > 0$ for μ_C -almost every x , then C is conical and rectifiable. (Stated on [6, p. 243], proved in proof of [6, Corollary 42.6].)

If C is rectifiable, then $\theta^m(C, x) > 0$ for μ_C -almost every x and hence C is again conical.

If $\theta^m(\mu_V, \cdot) \geq c > 0$ μ_V -almost everywhere then $\theta^m(C, \cdot) \geq c > 0$ μ_C -almost everywhere and C is conical (and rectifiable) [6, proof of Corollary 42.6].

Further note that if C is a tangent varifold from our example, then μ_C must be conical [6, 42.2 on p. 243].

For stationary 1-varifolds, the tangent varifolds C are always conical since μ_C is conical and $x \in S$ (equivalently, $p_{S^\perp}(x) = 0$) for all $(x, S) \in \text{spt } C$ [6, p. 243, l. 2–3].

¹ Since $C \neq 0$, we have $\theta^m(\mu_V, x_0) \in (0, \infty)$ from the Monotonicity formula for stationary varifolds, cf. [6, 40.5]. Therefore the assumptions of Corollary 42.6. (namely 42.1.) are satisfied.

2 Notation and definitions

For $0 \leq r < s \leq \infty$, denote by $S_r(\mathbb{R}^n)$ the sphere of radius r in \mathbb{R}^n and $A_r^s = A_r^s(\mathbb{R}^n) = \{x \in \mathbb{R}^n : r \leq \|x\| \leq s\}$ the annulus (or shell) in \mathbb{R}^n . Let $\mathbb{S}^1 = S_1(\mathbb{R}^2)$.

X denotes a smooth compactly supported vector field on \mathbb{R}^n (or on $\Omega \subset \mathbb{R}^n$).

If ν is a measure and M is ν -measurable then $\nu \llcorner M$ denotes the restriction of ν to M : $(\nu \llcorner M)(A) = \nu(M \cap A)$.

$\phi_\# \mu$ denotes the image measure [3, 2.1.2]:

$$\phi_\# \mu(A) = \mu(\phi^{-1}(A)). \quad (1)$$

If V is a k -varifold in \mathbb{R}^n (i.e., a measures on $G_k(\mathbb{R}^n)$, see Section 2.1), then we write $\phi_\# V$ for the image measure (if $\text{dom } \phi \subset G_k(\mathbb{R}^n)$) defined by (1) and

$$\phi_{\#\#} V$$

for the image varifold (assuming $\text{dom } \phi \subset \mathbb{R}^n$; see Section 2.4). The standard notation for both is the same ($\phi_\# V$) which would cause difficulties when reading some expressions in this paper.

2.1 Varifolds

To recall basic notions we follow and extend [4, p. 4–5, §Varifolds]. More details can be found in [1] and [6]. An m -varifold V on an open subset $\Omega \subset \mathbb{R}^n$ is a Radon measure on

$$G_m(\Omega) := \Omega \times G(n, m).$$

($G(n, m)$ denotes the *Grassmann manifold* consisting of m -dimensional linear subspaces of \mathbb{R}^n .) The space of m -varifolds is equipped with the *weak topology* given by saying that $V_i \rightarrow V$ if and only if $\int f dV_i \rightarrow \int f dV$ for all compactly supported, continuous real-valued functions on $G_m(\Omega)$. Varifolds can be combined using the addition which is addition of measures ($(c_1 V_1 + c_2 V_2)(B) = c_1 V_1(B) + c_2 V_2(B)$). A countable sum of varifolds is also a varifold, provided it is a Radon measure, i.e., it assign finite values to compact sets.

To a given m -varifold V , we associate a Radon measure μ_V on Ω by setting $\mu_V(A) = V(G_m(A))$ for $A \subset \Omega$. μ_V is called the *weight* of V ([6, p. 229]). As a partial converse, to a (Radon) m -rectifiable measure μ (see [4]) we can associate an m -rectifiable varifold $V = V_\mu$ by defining

$$V(B) = \mu\{x : (x, T_x) \in B\}, \quad B \subset G_m(\Omega) \quad (2)$$

where T_x is the approximate tangent plane at x .² If a countable sum of rectifiable varifolds is also a varifold then it is rectifiable. In this paper we need only the following

² Although there are several possible definition of approximate tangent plane (see [4], [1, p. 428, (3) and (b)] and [6, 11.2]), they agree μ -almost everywhere. The definitions of rectifiable varifolds in [1] and [4] essentially agree with that of [6], cf. footnote on [6, p. 77].

particular case of rectifiable varifolds (and their countable sums): $V = V_{c, \mathcal{H}^m \llcorner S}$ where $S = \text{range}(U)$ is a smooth parameterized surface and $c \in (0, \infty)$. Then the approximate tangent plane $T_{U(x)}$ agrees (μ_V -almost everywhere) with the classical tangent $\text{span}\{\partial U / \partial x^1, \dots, \partial U / \partial x^m\}$ to S , and V is exactly $c \cdot \mathbf{v}(S)$ from [1, p. 431].

The support of a measure μ is denoted by $\text{spt } \mu$. Note that if V is an m -varifold in $\Omega \subset \mathbb{R}^n$ then $\text{spt } V \subset G_m(\mathbb{R}^n)$ while $\text{spt } \mu_V \subset \mathbb{R}^n$. If V is an m -varifold (hence also a measure) and we say that V is *supported* by a set M if $M \subset \mathbb{R}^n$ and $\mu_V(\mathbb{R}^n \setminus M) = 0$ or $M \subset G_m(\mathbb{R}^n)$ and $V(G_m(\mathbb{R}^n) \setminus M) = 0$. If V is an m -varifold on $\Omega \subset \mathbb{R}^n$ and $M \subset \Omega$ then $V \llcorner G_m(M)$ might be called the *restriction* of V to M .

The *density* that we use in Introduction is defined as

$$\theta^k(\mu, x) = \lim_{r \rightarrow 0^+} \mu(A_0^r) / r^k$$

for a measure μ on \mathbb{R}^n , and by $\theta^k(V, x) = \theta^k(\mu_V, x)$ for a varifold V .

2.2 The first variation. Stationary varifolds. The mass. The curvature.

The *first variation* of an m -varifold V is a map from the space of smooth compactly supported vector fields on Ω to \mathbb{R} defined by (see [1, p. 434] and [6, p. 234, p. 51])

$$\delta V(X) = \int_{\Omega} \text{div}_S X(x) dV(x, S) \quad (3)$$

where $\text{div}_S X(x)$ is the divergence at x of the field X restricted (and projected) to affine subspace $x + S$ ([6, p. 234]). The idea is that the variation measures the rate of change in the 'size' (mass) of the varifold if it is perturbed slightly (see the alternate formula in [6, p. 233]). The *mass* of the varifold (see [6, p. 229]) is given by

$$\mathbf{M}(V) = V(G_m(\Omega)) = \mu_V(\Omega).$$

If $\delta V = 0$, then the varifold is said to be *stationary*. Varifold $V_{\mathcal{H}^m \llcorner S}$ associated to an m -dimensional affine plane S in \mathbb{R}^n is stationary.

Assume $V = V_{\mathcal{H}^m \llcorner S}$ is the rectifiable varifold associated to Hausdorff measure restricted to a smooth surface $S \subset \mathbb{R}^n$ such that the closure \bar{S} is a C^2 -smooth compact manifold with smooth $(m-1)$ -dimensional boundary $\partial S := \bar{M} \setminus M$. Then (3) reads

$$\delta V(X) = \int_S \text{div}_{T_x} X(x) d\mathcal{H}^m(x) \quad (4)$$

and can be (see [6, 7.6]) computed as

$$\delta V(X) = - \int_S X \cdot \mathbf{H} d\mathcal{H}^m - \int_{\partial S} X \cdot \eta d\mathcal{H}^{m-1} \quad (5)$$

where η is the inward pointing unit co-normal of ∂S , cf. [6, p. 43], and \mathbf{H} is the mean curvature vector ([6, 7.4]). If U is a parameterization of S and $\mathcal{B}(x) := \{\partial U / \partial x^1, \dots,$

$\partial U / \partial x^m$ happens to be orthonormal at x then \mathbf{H} can be obtained (cf. 7.4 and the last line on p. 44 of [6]) as

$$\mathbf{H}(U(x)) = \sum_{i=1}^m \left(\frac{\partial^2 U(x)}{(\partial x^i)^2} \right)^\perp$$

where v^\perp denotes orthogonal projection of v to the orthogonal complement of $T_{U(x)} = \text{span } \mathcal{B}(x)$. If $\mathcal{B}(x)$ is merely orthogonal at x , a linear change of variables $\tilde{x}_i = \sqrt{g^{ii}} x_i = \frac{1}{\|\partial U / \partial x^i\|} x_i$ reveals that

$$\mathbf{H}(U(x)) = \sum_{i=1}^m \left(g^{ii} \frac{\partial^2 U(x)}{(\partial x^i)^2} \right)^\perp. \quad (6)$$

We skip further derivations and note for the sake of completeness that (6) is in accordance with the following formula:

$$\mathbf{H}(U(x)) = \left(\sum_{ij} g^{ij} \frac{\partial^2 U}{\partial x^i \partial x^j} \right)^\perp \quad (7)$$

where (g^{ij}) is the inverse to the metric tensor (g_{ij}) (see [5, (1.11), p. 1098]).

2.3 An example

Exercise 2.1 Let H be a hyper-plane dividing \mathbb{R}^3 into two half-spaces H_1, H_2 . Let S_1, S_2 be 2-dimensional subspaces orthogonal to H . For $i = 1, 2$, let $V_i = (\mathcal{L}^3 \llcorner H_i) \times \delta_{S_i}$ (where δ_{S_i} is the Dirac measure at the point $S_i \in G(3, 2)$) and $V = V_1 + V_2$. Show that V is a stationary varifold on \mathbb{R}^3 .

Solution. From (3) and the divergence theorem we have $\delta V_i(X) = - \int_H x \cdot \eta_i d\mathcal{H}^2$ where η_i is the inward point unit normal to H_i . \square

Interpretation. V_i is the integral (or, uncountable “linear combination”) of varifolds $V_{i,x} = V_{\mathcal{H}^2 \llcorner (S_i + x)}$, $x \in S_i^\perp$. The variations $\delta V_{i,x}$ combine in the same way, and it turns out that the result is exactly opposite for V_1 and V_2 .

Remark 2.1 The varifold V is a 2-varifold supported by the 3-space $(\mu_V = \mathcal{L}^3, \text{spt } \mu_V = \mathbb{R}^3)$; V is non-rectifiable. V can be “approximated” by a rectifiable varifold supported by many half-planes touching H and parallel to S_1 (inside H_1) or S_2 (inside H_2). (The more half-planes, the better approximation and the less density on each of them.) This varifold cannot be stationary — the failure is located near H . There is a better “approximation” that is rectifiable and stationary, which is supported by strips of plane creating structure that branches and refines towards H . \square

Remark 2.2 Also the 2-varifold $(\mathcal{L}^3 \llcorner M_1) \times \delta_{S_1} + (\mathcal{L}^3 \llcorner M_2) \times \delta_{S_2}$ is stationary when $M_1 = \bigcup_{k \in \mathbb{Z}} [2k - 1, 2k] \times \mathbb{R} \times \mathbb{R}$, $M_2 = \bigcup_{k \in \mathbb{Z}} [2k, 2k + 1] \times \mathbb{R} \times \mathbb{R}$, $S_1 = \mathbb{R} \times \mathbb{R} \times \{0\}$, $S_2 = \mathbb{R} \times \{0\} \times \mathbb{R}$. \square

2.4 Tangents. Conical varifolds

For $x \in \mathbb{R}^n$ and $\lambda > 0$, let

$$\eta_{x,\lambda}(y) = \frac{y-x}{\lambda}, \quad y \in \mathbb{R}^n. \quad (8)$$

If V and C are m -varifolds on \mathbb{R}^n and $x \in \mathbb{R}^n$, we say that C is a *tangent varifold* to V at x , $C \in \text{VarTan}_x V$, if there exist $\lambda_i > 0$, $\lambda_i \rightarrow 0$ such that, for every continuous function f on $G_m(\mathbb{R}^n)$ with compact support,

$$\int f(y, S) dC(y, S) = \lim_{i \rightarrow \infty} (\lambda_i)^{-m} \int f(\eta_{x,\lambda_i}(y), S) dV(y, S).$$

This is equivalent to

$$\eta_{x,\lambda_i} \# V \rightarrow C$$

(weakly), which is the definition used in [6, p. 242-243].

The general definition of $\#$ for varifolds is (denoted differently by $\#$) in [6, p. 233] and it is slightly complicated. We need $\#$ only (i) with maps that are combination of translation and homothety, like (8), in which case

$$\eta_{x,\lambda_i} \# V(A) = (\lambda_i)^{-m} V(\{(y, S) : (\eta_{x,\lambda_i}(y), S) \in A\});$$

(ii) with orthonormal linear maps L , with

$$L \# V(A) = V(\{(y, S) : (L(y), L(S)) \in A\}).$$

An m -varifold C is *conical* if

$$\eta_{0,\lambda} \# C = C$$

for every $\lambda > 0$.

3 The non-rectifiable varifold

We start with an example of a non-rectifiable varifold, which is simpler. The rectifiable varifold in later sections is in fact a suitable rectifiable approximation of this non-rectifiable example. Thus, in this section we prove the following weaker version of Theorem 1.1.

Proposition 3.1 *There is a 2-varifold in \mathbb{R}^4 that has a non-conical (hence non-unique) tangent at a point. There is a 2-varifold in \mathbb{R}^4 that has a conical but non-unique tangent at a point.*

Proof The varifold will be supported by the three-dimensional surface^{3 4} in \mathbb{R}^4 parameterized by

$$F((a, b), (c, d)) = ace_1 + bce_2 + ade_3 + bde_4. \quad (9)$$

Then, for every $t > 0$,

$$F((ta, tb), (c, d)) = tF((a, b), (c, d)) = F((a, b), (tc, td)) \quad (10)$$

and

$$F((a, b), (c, d)) = c(ae_1 + be_2) + d(ae_3 + be_4) = a(ce_1 + de_3) + b(ce_2 + de_4). \quad (11)$$

Now, we are ready for an informal explanation of the idea. The surface is the union of a parameterized family of two-dimensional linear subspaces. In fact there is a pair of such representations that are “orthogonal”: We can fix $(a, b) \in \mathbb{S}^1$ as a parameter and use variables $(c, d) \in \mathbb{R}^2$ to create a 2-dimensional varifold $V_1^{(a,b)} := V_{\mathcal{H}^2 \llcorner \text{span}\{ae_1 + be_2, ae_3 + be_4\}}$ (which is stationary because it is associated to a 2-plane). Then we obtain a new (non-rectifiable) stationary varifold V_1 by averaging $V_1^{(a,b)}$ over all $(a, b) \in \mathbb{S}^1$. We also do the same with swapped (a, b) and (c, d) to obtain a different stationary varifold V_2 (yet with $\mu_{V_1} = \mu_{V_2}$). Suitable parts of the two varifolds can be joined together in similar way as in Exercise 2.1, with the separating hyperplane H replaced by a sphere. The resulting varifold is again stationary; the quantitative aspects of the formal proof of this fact depend on the presence of “orthogonality” of the parameterizations. Moreover, we can interleave an infinite number of concentric shells containing (parts of) V_1 and V_2 to obtain the target (non-rectifiable) varifold. Now we proceed with the formal definitions, arguments and calculations.

Let $0 \leq r < s \leq \infty$,

$$g_1((a, b), (c, d)) = \text{span}\{ae_1 + be_2, ae_3 + be_4\},$$

$$g_2((a, b), (c, d)) = \text{span}\{ce_1 + de_3, ce_2 + de_4\},$$

(g_i does not depend on all its parameters),

$$\begin{aligned} \phi_{1,r,s} &= (F, g_1): S_1(\mathbb{R}^2) \times A_r^s(\mathbb{R}^2) \rightarrow G_2(A_r^s(\mathbb{R}^4)), \\ &\quad (a, b, c, d) \mapsto (F((a, b), (c, d)), g_1((a, b), (c, d))), \end{aligned}$$

$$\begin{aligned} \phi_{2,r,s} &= (F, g_2): A_r^s(\mathbb{R}^2) \times S_1(\mathbb{R}^2) \rightarrow G_2(A_r^s(\mathbb{R}^4)), \\ &\quad (a, b, c, d) \mapsto (F((a, b), (c, d)), g_2((a, b), (c, d))), \end{aligned}$$

$$V_{1,r,s} = \phi_{1,r,s} \# (\mathcal{H}^1 \times \mathcal{L}^2), \quad (12)$$

$$V_{2,r,s} = \phi_{2,r,s} \# (\mathcal{L}^2 \times \mathcal{H}^1), \quad (13)$$

³ The surface is neither a linear space nor a convex set: it contains points $(1, 0, 0, 0)$ ($a = c = 1, b = d = 0$) and $(0, 0, 0, 1)$ ($a = c = 0, b = d = 1$) but does not contain $(1/2, 0, 0, 1/2)$. Indeed, $(t, 0, 0, t) = (ac, bc, ad, bd)$, $t \neq 0$ leads to $a \neq 0, c = t/a, b \neq 0, d = t/b$, then $bt/a = 0, at/b = 0$ and finally $b = 0 = a$, a contradiction.

⁴ The surface is actually a copy of the three-dimensional cone generated by $\mathbb{S}^1 \times \mathbb{S}^1$ as can be seen from the relation $(\cos \gamma, \sin \gamma, \cos \delta, \sin \delta) = (x + w, y - z, x - w, z + y)$ where $(x, y, z, w) = F((\cos \alpha, \sin \alpha), (\cos \beta, \sin \beta))$, $\gamma = \alpha - \beta, \delta = \alpha + \beta$. The surface was the first known nontrivial minimal cone in \mathbb{R}^4 , [5, p. 1113]. $\mathbb{S}^1 \times \mathbb{S}^1$ is so called Clifford torus. Recently, Simon Brendle announced that (up to a congruence) it is the only embedded minimal torus in \mathbb{S}^3 [2].

where $\#$ denotes the image of a measure (the image is a measure that happens to be a varifold), \mathcal{H}^1 is the one-dimensional Hausdorff measure in the unit sphere $S_1(\mathbb{R}^2)$ and \mathcal{L}^2 is the Lebesgue measure (on the annulus $A_r^s(\mathbb{R}^2) \subset \mathbb{R}^2$). From the definition of m -varifold we see that $V_{i,r,s}$ defined by (12), (13) are 2-varifolds. To see that $V_{i,r,s}$ can also be obtained by “averaging” (integrating, in the weak sense) 2-rectifiable varifolds, let

$$\begin{aligned}\phi_{1,r,s,(a,b)}(c,d) &= \phi_{1,r,s}((a,b), (c,d)), & (c,d) &\in A_r^s(\mathbb{R}^2), \\ \phi_{2,r,s,(c,d)}(a,b) &= \phi_{2,r,s}((a,b), (c,d)), & (a,b) &\in A_r^s(\mathbb{R}^2),\end{aligned}$$

$$V_{1,r,s,(a,b)} := \phi_{1,r,s,(a,b)} \# \mathcal{L}^2 \stackrel{*}{=} V_{\mathcal{H}^2 \llcorner (\text{span}\{ae_1+be_2, ae_3+be_4\} \cap A_r^s(\mathbb{R}^4))}, \quad (14)$$

$$V_{2,r,s,(c,d)} := \phi_{2,r,s,(c,d)} \# \mathcal{L}^2 \stackrel{*}{=} V_{\mathcal{H}^2 \llcorner (\text{span}\{ce_1+de_3, ce_2+de_4\} \cap A_r^s(\mathbb{R}^4))}, \quad (15)$$

where “ $\stackrel{*}{=}$ ” are valid under condition $(a,b) \in \mathbb{S}^1$ or $(c,d) \in \mathbb{S}^1$, respectively. Then, by the Fubini theorem,

$$V_{1,r,s} = \int_{(a,b) \in \mathbb{S}^1} V_{1,r,s,(a,b)} d\mathcal{H}^1, \quad (16)$$

$$V_{2,r,s} = \int_{(c,d) \in \mathbb{S}^1} V_{2,r,s,(c,d)} d\mathcal{H}^1. \quad (17)$$

Since $V_{i,r,s}(\cdot, \cdot)$ is just the varifold corresponding to an annulus part of a 2-plane ($\mathbf{H} = 0$), its first variation corresponds to the inward pointing unit co-normal field supported on the two circles (cf. (5)):

$$\begin{aligned}\delta V_{1,r,s,(a,b)}(X) &= \int_{\{F(a,b,c,d): c^2+d^2=s^2\}} X \cdot N d\mathcal{H}^1 - \int_{\{F(a,b,c,d): c^2+d^2=r^2\}} X \cdot N d\mathcal{H}^1, \\ \delta V_{2,r,s,(c,d)}(X) &= \int_{\{F(a,b,c,d): a^2+b^2=s^2\}} X \cdot N d\mathcal{H}^1 - \int_{\{F(a,b,c,d): a^2+b^2=r^2\}} X \cdot N d\mathcal{H}^1\end{aligned}$$

where $N(x) = x/\|x\|$ and where we leave out the first term if $s = \infty$. The second term is zero if $r = 0$. Integrating over $(a,b) \in S_1(\mathbb{R}^2)$, respective over $(c,d) \in S_1(\mathbb{R}^2)$, and changing the variables back to the image of F (where it becomes a circle of radius s or r), we get

$$\begin{aligned}\delta V_{1,r,s}(X) &= \int_{F(S_1(\mathbb{R}^2) \times S_s(\mathbb{R}^2))} X \cdot N d\frac{\mathcal{H}^1}{s} \times \mathcal{H}^1 - \int_{F(S_1(\mathbb{R}^2) \times S_r(\mathbb{R}^2))} X \cdot N d\frac{\mathcal{H}^1}{r} \times \mathcal{H}^1 \\ \delta V_{2,r,s}(X) &= \int_{F(S_s(\mathbb{R}^2) \times S_1(\mathbb{R}^2))} X \cdot N d\mathcal{H}^1 \times \frac{\mathcal{H}^1}{s} - \int_{F(S_r(\mathbb{R}^2) \times S_1(\mathbb{R}^2))} X \cdot N d\mathcal{H}^1 \times \frac{\mathcal{H}^1}{r}\end{aligned}$$

(Again, if $s = \infty$ or $r = 0$, the first or second term has to be replaced by zero. In particular, $V_{i,0,\infty}$ are stationary.) Therefore $V_{1,r,s}$ and $V_{2,r,s}$ have the same first variation, $\delta V_{1,r,s} = \delta V_{2,r,s}$, and (cf. (3))

$$\int_{\Omega} \text{div}_S X(x) dV_{1,r,s}(x, S) = \int_{\Omega} \text{div}_S X(x) dV_{2,r,s}(x, S). \quad (18)$$

We show that

$$V = V_{\{r_i\}_{i \in \mathbb{Z}}} = \sum_{i=-\infty}^{\infty} (V_{1,r_{2i},r_{2i+1}} + V_{2,r_{2i+1},r_{2i+2}}) \quad (19)$$

is a stationary varifold for any increasing sequence $\{r_i\}_{i \in \mathbb{Z}}$ with $\lim_{i \rightarrow \infty} r_i = \infty$ and $\lim_{i \rightarrow -\infty} r_i = 0$.

Indeed, V is a Radon measure on $G_2(\mathbb{R}^4)$ since, e.g., $V(G_2(A_0^s)) = \pi \cdot \pi s^2$. Using (3) and substituting from (18)

$$\begin{aligned} \delta V(X) &= \sum_{i=-\infty}^{\infty} \int_{\Omega} \operatorname{div}_S X(x) dV_{1,r_{2i},r_{2i+1}} + \int_{\Omega} \operatorname{div}_S X(x) dV_{2,r_{2i+1},r_{2i+2}} \\ &= \sum_{i=-\infty}^{\infty} \int_{\Omega} \operatorname{div}_S X(x) dV_{1,r_i,r_{i+1}} = \int_{\Omega} \operatorname{div}_S X(x) dV_{1,0,\infty} = 0 \end{aligned}$$

since $V_{1,0,\infty}$ is stationary.

If $r_i = 2^i$ then $C := V$ is a non-conical tangent varifold to V at $0 \in \mathbb{R}^4$. (Also $\eta_{0,\lambda} \# V \in \operatorname{VarTan}_0 V$ for $\lambda \in (0, \infty)$.)

If $r_i = 2^{2^i}$ then $V_{1,0,\infty}$ and $V_{2,0,\infty}$ are two different conical tangent varifolds to V at $0 \in \mathbb{R}^4$. (Also $V_{1,0,r} + V_{2,r,\infty} \in \operatorname{VarTan}_0 V$ and $V_{2,0,r} + V_{1,r,\infty} \in \operatorname{VarTan}_0 V$ for $r \in (0, \infty)$.)

The above statements about “non-conical” tangent and about “two different” varifolds need a bit of justification and deserve to be formulated separately. \square

Lemma 3.1 *Varifold $V = V_{\{r_i\}_{i \in \mathbb{Z}}}$ from (19) is not conical. Furthermore, $V_{1,r,s} \neq V_{2,r,s}$ for any $0 \leq r < s \leq \infty$.*

Proof We claim that

$$\text{if } F(x) = F(y) \neq 0 \text{ then } g_1(x) \neq g_2(y). \quad (20)$$

For $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \setminus \{0\}$, we have $F^{-1}(x) = \emptyset$ or

$$F^{-1}(x) \subset \{(\pm t \sqrt{x_1^2 + x_3^2}, \pm t \sqrt{x_2^2 + x_4^2}, \pm \frac{\|x\|}{t} \sqrt{x_1^2 + x_2^2}, \pm \frac{\|x\|}{t} \sqrt{x_3^2 + x_4^2}) : t > 0\}.$$

First we show that

$$S_1 := g_1((a, b), (c, d)) \text{ is different from } S_2 := g_2((\pm a, \pm b), (\pm c, \pm d)) \quad (21)$$

apart from singular cases $a = b = 0$ or $c = d = 0$ (when $F((a, b), (c, d)) = 0$): Since g_2 does not depend on a and b , we have $S_2 = g_2((a, b), (\pm c, \pm d))$. Since g_1 does not depend on c and d , we can freely change the sign of c (and d) in (21). Therefore it is enough to consider $S_2 = g_2((a, b), (c, d))$. Assume that $a^2 + b^2 \neq 0$ and $c^2 + d^2 \neq 0$. S_1 and S_2 are two-dimensional subspaces and if $S_1 = S_2$ then $\operatorname{span}(S_1 \cup S_2)$ is two-dimensional as well, i.e., the matrix

$$\begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & a & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

has rank 2. Then $(a^2 + b^2)c = -\begin{vmatrix} a & b & 0 \\ 0 & 0 & a \\ 0 & c & 0 \end{vmatrix} + \begin{vmatrix} a & b & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{vmatrix} = 0 + 0 = 0$, $(a^2 + b^2)d = \begin{vmatrix} a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & d \end{vmatrix} - \begin{vmatrix} b & 0 & 0 \\ 0 & a & b \\ 0 & d & 0 \end{vmatrix} = 0$. Hence $c = 0, d = 0$, a contradiction showing that (21) is true.

Since $g_i((ta, tb), (uc, ud)) = g_i((a, b), (c, d)) = S_i$ for $i = 1, 2$ and $t, u \in \mathbb{R} \setminus \{0\}$, we get (20).

By (20), $V_{1,r,s}$ and $V_{2,r,s}$ are supported by disjoint subsets of $G_2(\mathbb{R}^4)$ whenever $r > 0$. (For $r = 0$, $\text{spt } V_{1,r,s} \cap \text{spt } V_{2,r,s} \subset \{(0, 0, 0, 0)\} \times G(4, 2)$.) Hence $V_{1,r,s} \neq V_{2,r,s}$ for any $0 \leq r < s \leq \infty$. Obviously, varifold $V = V_{\{r_i\}_{i \in \mathbb{Z}}}$ is not conical. \square

4 The rectifiable varifold

The rectifiable stationary varifold (lets call it V_{rect} for now) will be obtained as a suitable approximation of the above non-rectifiable V . Instead of planar strips smoothed out by averaging, the support consists now of pieces (“rings”) of curved surface that will branch towards the boundary of the layer.

Since now the pieces of V_{rect} are not oriented radially ($x \notin S$ for many $(x, S) \in \text{spt } V_{\text{rect}}$), the ratio $V_{\text{rect}}(G_2(B(0, r)))/r^2$ necessarily decreases as r decreases. (This is a corollary to the Monotonicity formula [6, 17.5].) Therefore we have, and do, take special care to make sure that the density $\theta^2(V_{\text{rect}}, 0)$ does not vanish.

The proof continues towards the end of this paper and depends on the calculations summarized in the following lemmata.

The varifold will be again supported by the three-dimensional surface parameterized by F , see (9).

In every point of $x \in \text{range } F \setminus \{0\}$, we will frequently refer to the radial direction $N(x) = x/\|x\|$ and to a selected tangential direction. The latter is conveniently expressed by matrix multiplication.

Let J_{13}^{24} be the matrix that rotates $e_1 \rightarrow e_3$ and $e_2 \rightarrow e_4$ given by

$$J_{13}^{24} = J(0, 1) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (22)$$

For $\varepsilon \geq 0$ and $x \in \mathbb{R}^4 \setminus \{0\}$, let

$$G_{\text{rad} \& J_{13}^{24}}^\varepsilon(x) = \{\text{span}\{u, v\} : \{u, v\} \text{ orthonormal}, \\ \|u - N(x)\| \leq \varepsilon, \|v - J_{13}^{24}N(x)\| \leq \varepsilon\}, \quad (23)$$

and

$$G_{\text{rad} \& J_{13}^{24}}^\varepsilon = \{(x, S) : x \in \mathbb{R}^4 \setminus \{0\}, S \in G_{\text{rad} \& J_{13}^{24}}^\varepsilon(x)\}. \quad (24)$$

Then $\{G_{\text{rad} \& J_{13}^{24}}^\varepsilon(x) : \varepsilon > 0\}$ is a neighborhood base for a special point $\text{span}\{N(x), J_{13}^{24}N(x)\} \in G(4, 2)$, which is the span of the radial direction and the direction determined by J_{13}^{24} . From this comes the subscript in our notation.

Note that if we let

$$J_{12}^{34} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (25)$$

and define $G_{\text{rad} \& J_{12}^{34}}^\varepsilon(x)$ accordingly then there is $\varepsilon_0 > 0$ (independent of x) such that

$$G_{\text{rad} \& J_{13}^{24}}^\varepsilon(x) \cap G_{\text{rad} \& J_{12}^{34}}^\varepsilon(x) = \emptyset \quad (26)$$

for all $\varepsilon \in [0, \varepsilon_0)$. To see that, it is only needed to observe that

$$\text{span}\{N(x), J_{13}^{24}N(x)\} \neq \text{span}\{N(x), J_{12}^{34}N(x)\}. \quad (27)$$

This is similar to (21) but now the proof is even easier. First consider (27) in the special case when $N(x) = e_1$. Then if (27) were not valid, then the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

would have range at most two. Using a rotation, we see that (27) in general is equivalent to (27) in the special case when $N(x) = e_1$.

4.1 Basic surface, rings and their joins.

Lemma 4.1 1. Let $d > 0$, $\alpha_0 \in \mathbb{R}$ and

$$r(\alpha) = r^{d, \alpha_0}(\alpha) = \sqrt{d / \cos 2(\alpha - \alpha_0)} \quad \text{for all } \alpha - \alpha_0 \in (-\pi/4, \pi/4).$$

Consider the parameterized surface $U(\alpha, \beta) = U^{d, \alpha_0}(\alpha, \beta)$,

$$U(\alpha, \beta) = (r(\alpha) \cos \alpha \cos \beta, r(\alpha) \sin \alpha \cos \beta, r(\alpha) \cos \alpha \sin \beta, r(\alpha) \sin \alpha \sin \beta),$$

$$\alpha - \alpha_0 \in (-\pi/4, \pi/4), \beta \in \mathbb{R}.$$

(U is 2π -periodic in β , and injective on every period.) Then U is a minimal surface.

2. Let

$$S_{t_1, t_2} = S_{t_1, t_2}^{d, \alpha_0} = \{U(\alpha, \beta) : \alpha \in (t_1, t_2), \beta \in \mathbb{R}\}, \quad (\text{the “ring”}) \quad (28)$$

$$S_{t_1} = \{U(t_1, \beta) : \beta \in \mathbb{R}\}, \quad (29)$$

$$S = S_{\alpha_0 - \pi/4, \alpha_0 + \pi/4} = \text{range}(U).$$

Then the rectifiable varifold $V_{\mathcal{H}^2 \llcorner S}$ is stationary.

3. (The ring varifold and its first variation.) For every $x \in S$, find any p satisfying $U(p) = x$ and let

$$\boldsymbol{\eta}_{\alpha_0}(x) = N(\partial U(p) / \partial \alpha)$$

where $N(y) = y / \|y\|$.

For $\alpha_0 - \pi/4 < t_1 < t_2 < \alpha_0 + \pi/4$, let

$$V_{t_1, t_2}^{d, \alpha_0} = V_{\mathcal{H}^2 \llcorner S_{t_1, t_2}^{d, \alpha_0}}. \quad (30)$$

Then,

$$\delta V_{t_1, t_2}^{d, \alpha_0}(X) = \int_{S_{t_2}} X(x) \cdot \boldsymbol{\eta}_{\alpha_0}(x) d\mathcal{H}^1 - \int_{S_{t_1}} X(x) \cdot \boldsymbol{\eta}_{\alpha_0}(x) d\mathcal{H}^1. \quad (31)$$

4. (Two rings at touch.) If $\alpha_1 \leq \alpha \leq \alpha_2$ and $\alpha - \alpha_1 = \alpha_2 - \alpha \in [0, \pi/4]$ then

$$U^{d, \alpha_1}(\alpha, \beta) = U^{d, \alpha_2}(\alpha, \beta) \quad (32)$$

and

$$\eta_{\alpha_1}(U^{d, \alpha_1}(\alpha, \beta)) - \eta_{\alpha_2}(U^{d, \alpha_2}(\alpha, \beta)) = 2 \sin 2(\alpha - \alpha_1) \cdot N(U^{d, \alpha_1}(\alpha, \beta)) \quad (33)$$

is a radial vector at the point.

5. The tangent plane to $U = U^{d, \alpha_0}$ at $x = U(\alpha, \beta)$ belongs to $G_{\text{rad} \& J_{13}^{24}}^{2 \cos 2(\alpha - \alpha_0)}(x)$ and

$$\text{spt } V_{t_1, t_2}^{d, \alpha_0} \subset G_{\text{rad} \& J_{13}^{24}}^\varepsilon$$

where $\varepsilon = 2 \max \cos([2(t_1 - \alpha_0), 2(t_2 - \alpha_0)])$.

6. (Mass distribution)

$$\mathbf{M}(V_{t_1, t_2}^{d, \alpha_0}) = \pi d (\tan 2(t_2 - \alpha_0) - \tan 2(t_1 - \alpha_0)).$$

For every $0 < \sqrt{d} \leq r_1 < r_2$ there is a number $\rho = \rho(d, r_1, r_2) \in [r_1, r_2]$ such that whenever $\alpha_0 < t_1 < t_2 < \alpha_0 + \pi/4$, and $t_1 \leq s_1 \leq s_2 \leq t_2$ then,

$$\begin{aligned} \mathbf{M}(V_{t_1, t_2}^{d, \alpha_0}) &= \pi \left| \sqrt{r(t_2)^4 - d^2} - \sqrt{r(t_1)^4 - d^2} \right| \\ &= \frac{1}{\sqrt{1-d^2/(\rho(d, r(t_1), r(t_2)))^4}} \pi |r(t_2)^2 - r(t_1)^2| \end{aligned}$$

and

$$\begin{aligned} V_{t_1, t_2}^{d, \alpha_0}(G_2(A_{r(s_1)}^{r(s_2)})) &= \mathbf{M}(V_{s_1, s_2}^{d, \alpha_0}) = \pi \left| \sqrt{r(s_2)^4 - d^2} - \sqrt{r(s_1)^4 - d^2} \right| \\ &= \frac{1}{\sqrt{1-d^2/(\rho(d, r(s_1), r(s_2)))^4}} \pi |r(s_2)^2 - r(s_1)^2|. \end{aligned}$$

If $\alpha_0 - \pi/4 < t_1 < t_2 < \alpha_0$, the same holds with $A_{r(s_1)}^{r(s_2)}$ replaced by $A_{r(s_2)}^{r(s_1)}$ and ρ extended by formula $\rho(d, r_1, r_2) := \rho(d, r_2, r_1)$ for $\sqrt{d} \leq r_2 < r_1$.

Remark 4.1 Our construction is related to the construction of Section 3: We have $S = \text{range } U \subset \text{range } F$ where F is as in (9), (10). In fact, $U(\alpha, \beta) = r(\alpha)F((\cos \alpha, \sin \alpha), (\cos \beta, \sin \beta))$. For $\alpha_0 + \pi/4 - \varepsilon < t_1 < t_2 < \alpha_0 + \pi/4$ (or analogously for $\alpha_0 - \pi/4 < t_1 < t_2 < \alpha_0 - \pi/4 + \varepsilon$), and $r = r(t_1)$, $s = r(t_2)$, the ring S_{t_1, t_2} is intended to be a perturbation of the annulus supporting $V_{1, r, s, (\cos t_1, \sin t_1)}$ from (14); the relation can be better seen if α is considered as a function of the radius: if $\alpha(\rho) = r^{-1}(\rho)$ then $U(\alpha(\rho), \beta) = \rho F((\cos \alpha(\rho), \sin \alpha(\rho)), (\cos \beta, \sin \beta))$ and $\alpha'(\rho)$ is small provided $\rho \gg \sqrt{d}$ (α is near $\alpha_0 \pm \pi/4$). \square

We will give two arguments for the minimality of surface U , the first one is easy but slightly incomplete: Let $\alpha_0 - \pi/4 < t_1 < t_2 < \alpha_0 + \pi/4$ with t_2 close to t_1 , and consider the part of the surface determined by a range $t \in (t_1, t_2)$ (cf. (28)); this is the surface created by a certain “rotation” from curve

$$\gamma(t) := (r(t)\cos t, r(t)\sin t, 0, 0), \quad t \in (t_1, t_2).$$

The boundary of the selected part consists of two circles S_{t_1}, S_{t_2} (see (29)). To this correspond fixed values $\gamma(t_1), \gamma(t_2)$, as boundary conditions for γ .

Our first and incomplete argument for the minimality of U is based on comparing the area of the selected part of U with surfaces corresponding to other possible curves γ in $\mathbb{R}^2 \times \{0\}^2$ with the same boundary condition.

The area is given by the formula

$$A = 2\pi \int_{(t_1, t_2)} \|\gamma'(t)\| \cdot \|\gamma(t)\| dt$$

since the length of the circle through $\gamma(t)$ is $2\pi\|\gamma(t)\|$. We will view γ as a curve in $\mathbb{R}^2 \cong \mathbb{R}^2 \times \{0\}^2$, and assume that γ is the graph of a function r in polar coordinates, that is $\gamma(t) = (r(t)\cos t, r(t)\sin t)$. On $\mathbb{R}^2 = \mathbb{C}$, consider the map $z \mapsto z^2$ whose derivative is $2z$. That maps curve γ to a curve γ^2 (where $\gamma^2(t) = (\gamma(t))^2 \in \mathbb{C}$) whose length

$$L = \int_{(t_1, t_2)} \|(\gamma^2)'(t)\| dt = \int_{(t_1, t_2)} 2\|\gamma'(t)\| \cdot \|\gamma(t)\| dt$$

we find to be directly proportional to A . It is well known that L is minimal if γ^2 is the segment connecting its endpoints. A special case is a vertical segment given in polar coordinates by $(\tilde{r}, \tilde{\alpha})$ with $\tilde{r} = d / \cos \tilde{\alpha}$; the general case is $\tilde{r} = d / \cos(\tilde{\alpha} - \tilde{\alpha}_0)$. Since $z \mapsto z^2$ is expressed in polar coordinates as $(r, \alpha) \mapsto (\tilde{r}, \tilde{\alpha}) = (r^2, 2\alpha)$, we obtain the curve $\gamma(t) = (r(t)\cos t, r(t)\sin t)$ with $r(t) = \sqrt{d / \cos 2(t - \alpha_0)}$, $t \in [t_1, t_2]$. The corresponding rotation surface is our best candidate for the minimum area surface spanned between S_{t_1} and S_{t_2} and U likely is a minimal surface.

Proof of Lemma 4.1

1. For formal verification of the minimality of surface U , it is enough to verify that $\mathbf{H}(U) = 0$.

For $a, b, \alpha, \beta \in \mathbb{R}$, let

$$B = B(\beta) = J(\cos \beta, \sin \beta), \quad \text{where} \quad J(a, b) = \begin{pmatrix} a & 0 & -b & 0 \\ 0 & a & 0 & -b \\ b & 0 & a & 0 \\ 0 & b & 0 & a \end{pmatrix}$$

and (since we choose to treat the vectors, including U , as column vectors, we will distinguish that in notation from this moment)

$$A = A(\alpha) = (\cos \alpha, \sin \alpha, 0, 0)^T.$$

Then

$$U = rBA$$

where r is a function of α :

$$U(\alpha, \beta) = r(\alpha)B(\beta)A(\alpha), \quad \alpha \in (-\pi/4, \pi/4), \beta \in \mathbb{R}.$$

Note that obviously $\|U\| = r$, hence

$$N(U) = BA.$$

We have

$$\frac{\partial U}{\partial \alpha} = r'BA + rBA' = B(r'A + rA') \quad (34)$$

$$\frac{\partial U}{\partial \beta} = rB'A \quad (35)$$

where

$$A' = (-\sin \alpha, \cos \alpha, 0, 0)^T, \quad B' = J(-\sin \beta, \cos \beta).$$

Furthermore,

$$A'' = (-\cos \alpha, -\sin \alpha, 0, 0)^T = -A, \quad B'' = J(-\cos \beta, -\sin \beta) = -B$$

and hence

$$\frac{\partial^2 U}{\partial \alpha^2} = B(r''A + 2r'A' + rA'') = B((r'' - r)A + 2r'A') \quad (36)$$

$$\frac{\partial^2 U}{\partial \beta^2} = rB''A = -rBA \quad (37)$$

Obviously

$$A^T A = (A')^T A' = 1 \quad A^T A' = (A')^T A = 0. \quad (38)$$

It is immediate that $J(a, b)^T = J(a, -b)$ and $J(a, b)^T J(a, b) = (a^2 + b^2)I$ where I is the identity matrix; in particular

$$B^T B = I, \quad (39)$$

$$(B')^T B' = I. \quad (40)$$

Hence

$$B^{-1} = B^T. \quad (41)$$

Furthermore, $J(b, a)J(a, b) = J(0, a^2 + b^2)$, in particular

$$B^T B' = J(0, 1) \quad (42)$$

and $B'B^T = J(0, 1)$. Multiplying that by B from the right (see (41)) we get

$$B' = J(0, 1)B. \quad (43)$$

The metric tensor is

$$g_{11} = \frac{\partial U}{\partial \alpha} \cdot \frac{\partial U}{\partial \alpha} = (r'A + rA')^T B^T B (r'A + rA') \quad (44)$$

$$\stackrel{(39)}{=} (r'A + rA')^T (r'A + rA') \stackrel{(38)}{=} (r')^2 + r^2$$

$$g_{22} = \frac{\partial U}{\partial \beta} \cdot \frac{\partial U}{\partial \beta} = rA^T (B')^T rB'A$$

$$\stackrel{(40)}{=} r^2 A^T A \stackrel{(38)}{=} r^2$$

$$g_{12} = g_{21} = \frac{\partial U}{\partial \alpha} \cdot \frac{\partial U}{\partial \beta} = (r'A + rA')^T B^T rB'A \quad (45)$$

$$\stackrel{(42)}{=} r(r'A + rA')^T J(0, 1)A = 0$$

since $A, A' \in \mathbb{R}^2 \times \{0\}^2$ while $J(0, 1)A \in \{0\}^2 \times \mathbb{R}^2$. Therefore

$$(g_{ij}) = \begin{pmatrix} (r')^2 + r^2 & 0 \\ 0 & r^2 \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} \frac{1}{(r')^2 + r^2} & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}.$$

We want to verify $\mathbf{H}(U) = 0$ using (6) (or, equivalently, (7)). Thus we want to verify

$$v^\perp = 0, \quad \text{that is,} \quad v \in \text{span} \left\{ \frac{\partial U}{\partial x^i} \right\}$$

where

$$v = \frac{1}{(r')^2 + r^2} B((r'' - r)A + 2r'A') + \frac{1}{r^2} (-rBA).$$

That is

$$\frac{1}{(r')^2 + r^2} B((r'' - r)A + 2r'A') + \frac{1}{r^2} (-rBA) \in \text{span}\{B(r'A + rA'), rB'A\}.$$

Multiplying by B^{-1} and using (41), (42), we get equivalent relation

$$\frac{1}{(r')^2 + r^2} ((r'' - r)A + 2r'A') - \frac{1}{r} A \in \text{span}\{r'A + rA', rJ(0, 1)A\}.$$

Since $A, A' \in \mathbb{R}^2 \times \{0\}^2$, while $rJ(0, 1)A \in \{0\} \times \mathbb{R}^2$, the latter can be removed:

$$\frac{1}{(r')^2 + r^2} ((r'' - r)A + 2r'A') - \frac{1}{r} A \in \text{span}\{r'A + rA'\}.$$

Now the relation reduces to $\mathbb{R}^2 \times \{0\}^2$, where A, A' form an orthogonal base. We have $r'A + rA' \perp rA - r'A'$ and our relation is equivalent to

$$(rA - r'A')^T \left(\frac{1}{(r')^2 + r^2} ((r'' - r)A + 2r'A') - \frac{1}{r} A \right) = 0.$$

Using (38) this reduces to

$$rr'' - 3(r')^2 - 2r^2 = 0.$$

It is easy to check that our function $r(\alpha) = \sqrt{d / \cos 2(\alpha - \alpha_0)}$ verifies this equation.

Thus we proved that the mean curvature vector $\mathbf{H}(U)$ is identically zero and $U(\alpha, \beta)$ is a minimal surface.

2. Since $\mathbf{H}(U) = 0$ and there is no boundary (U is defined on \mathbb{R}^2 and essentially injective) the associated varifold is stationary.

3. To obtain (31), it is enough to use (5); The boundary of $S = S_{t_1, t_2}^{d, \alpha_0}$ is $S_{t_1} \cup S_{t_2}$, and if $U(p) \in \partial S$ then $\partial U(p)/\partial \beta$ is obviously tangent to ∂S and $\eta := \partial U(p)/\partial \alpha$ is orthogonal to it, see (45). If $p = (t_1, \beta)$ then η is an inner normal, if $p = (t_2, \beta)$ then it is outer.

4. Assume now that $\alpha_1 \leq \alpha \leq \alpha_2$ and

$$\alpha - \alpha_1 = \alpha_2 - \alpha \in [0, \pi/4]. \quad (46)$$

Then $r^{d, \alpha_1}(\alpha) = r^{d, \alpha_2}(\alpha)$ and hence $U^{\alpha_0}(\alpha, \beta) = U^{\alpha_1}(\alpha, \beta)$.

At any point (α, β) satisfying (46) we have, by (34) and (44),

$$\begin{aligned} \frac{\partial U}{\partial \alpha} &= r'BA + rBA' \\ N\left(\frac{\partial U}{\partial \alpha}\right) &= \frac{r'}{\sqrt{r'^2 + r^2}}BA + \frac{r}{\sqrt{r'^2 + r^2}}BA' \end{aligned} \quad (47)$$

where A, B and r are the same regardless if U^{d, α_1} or U^{d, α_2} is considered. Only r' is different:

$$(r^{d, \alpha_1})'(\alpha) = -(r^{d, \alpha_2})'(\alpha).$$

Letting, e.g., $\alpha_0 := \alpha_1$, we have

$$r = \sqrt{d} \cos^{-1/2} 2(\alpha - \alpha_0) \quad (48)$$

$$\begin{aligned} r' &= \sqrt{d} \cos^{-3/2} 2(\alpha - \alpha_0) \sin 2(\alpha - \alpha_0) \\ \sqrt{r'^2 + r^2} &= \sqrt{d} \cos^{-3/2} 2(\alpha - \alpha_0). \end{aligned} \quad (49)$$

Since (47) are the values of η_{α_1} and η_{α_2} , we get (33), that is,

$$\eta_{\alpha_1}(U^{d, \alpha_1}(\alpha, \beta)) - \eta_{\alpha_2}(U^{d, \alpha_2}(\alpha, \beta)) = cBA = cN(U^{d, \alpha_1}(\alpha, \beta))$$

where

$$c = \frac{2r'}{\sqrt{r'^2 + r^2}} = 2 \sin 2(\alpha - \alpha_0).$$

To prove 5., it is enough to show that the tangent to U at $U(\alpha, \beta)$ is the plane spanned by orthonormal base $\{N(\frac{\partial U}{\partial \alpha}(\alpha, \beta)), N(\frac{\partial U}{\partial \beta}(\alpha, \beta))\}$ where

$$\left\| N\left(\frac{\partial U}{\partial \alpha}(\alpha, \beta)\right) - N(U(\alpha, \beta)) \right\| \leq 2 \cos 2(\alpha - \alpha_0) \quad (50)$$

$$N\left(\frac{\partial U}{\partial \beta}(\alpha, \beta)\right) = J(0, 1)U(\alpha, \beta). \quad (51)$$

The two vectors are orthogonal by (45). Using $N(U) = BA$ and (47) we get

$$\|N(\frac{\partial U}{\partial \alpha}) - N(U)\| \leq \frac{2r}{\sqrt{r'^2 + r^2}} = 2 \cos 2(\alpha - \alpha_0) \quad (52)$$

which is (50). Furthermore we have

$$N\left(\frac{\partial U}{\partial \beta}(\alpha, \beta)\right) = \frac{1}{\sqrt{g_{22}}} \frac{\partial U(\alpha, \beta)}{\partial \beta} \stackrel{(35)}{=} B'A \stackrel{(43)}{=} J(0, 1)BA = J(0, 1)U(\alpha, \beta)$$

which is (51).

6. The mass formula is directly obtained by integration. Since $g_{12} = 0$, the 2-volume element has a simple form.

$$\begin{aligned} \mathbf{M}(V_{t_1, t_2}^{d, \alpha_0}) &= \mathcal{H}^2 S_{t_1, t_2}^{d, \alpha_0} = \int_{[t_1, t_2]} d\alpha \int_{[0, 2\pi]} d\beta \sqrt{g_{11} g_{22}} \\ &= 2\pi \int_{[t_1, t_2]} d\alpha r \sqrt{r'^2 + r^2} \stackrel{(49)}{=} 2\pi \int_{[t_1, t_2]} d\alpha d \cos^{-2} 2(\alpha - \alpha_0) \\ &= \pi d (\tan 2(t_2 - \alpha_0) - \tan 2(t_1 - \alpha_0)) \end{aligned}$$

If $\alpha_0 \notin [t_1, t_2]$ then $\operatorname{sgn} \tan 2(t_2 - \alpha_0) = \operatorname{sgn} \tan 2(t_1 - \alpha_0)$ and

$$d |\tan 2(t_i - \alpha_0)| = \sqrt{\frac{d^2}{\cos^2 2(t_i - \alpha_0)} - d^2} = \sqrt{r(t_i)^4 - d^2}$$

since $r(t_i)^2 = d / \cos 2(t_i - \alpha_0)$. This gives the mass in the form

$$\pi \left| \sqrt{r(t_2)^4 - d^2} - \sqrt{r(t_1)^4 - d^2} \right|.$$

The expression that contains ρ is obtained by the Mean value theorem applied to function $q \mapsto \sqrt{q^2 - d^2}$ on interval $[r(t_1)^2, r(t_2)^2]$ or $[r(t_2)^2, r(t_1)^2]$. (Thus ρ depends on d , $r(t_1)$ and $r(t_2)$ but, naturally, not on α_0 .) Since obviously $S_{t_1, t_2} \cap A_{r(s_1)}^{r(s_2)} = S_{s_1, s_2}$ we have

$$V_{t_1, t_2}^{d, \alpha_0}(G_2(A_{r(s_1)}^{r(s_2)})) = \mathbf{M}(V_{s_1, s_2}^{d, \alpha_0}).$$

□

4.2 Mini-layer. Details about branching.

From the ring varifolds we construct two types of (mini-layer) varifolds: V_1 branching inwards and V_2 branching outwards. That is, δV_1 is supported on a number of circles of larger radius and twice as much circles of smaller radius. We carefully compute the densities of δV_i on the circles and record the mass distribution.

Lemma 4.2 For $\rho > 0$ and $\alpha \in \mathbb{R}$ denote

$$S(\rho, \alpha) = \{\rho(\cos \alpha \cos \beta, \sin \alpha \cos \beta, \cos \alpha \sin \beta, \sin \alpha \sin \beta) : \beta \in \mathbb{R}\}. \quad (53)$$

Let $k \in \mathbb{N}$, $k > 20$ and $\gamma \in (\pi/8, \pi/4)$ be fixed. Let

$$\begin{aligned} \sigma &= \sigma_{k, \gamma} = \sqrt{\frac{\cos 2\gamma}{\cos 2(\gamma - \pi/k)}} \in (0, 1), \\ \varepsilon &= 2 \cos 2(\gamma - \pi/k) \end{aligned}$$

and

$$\begin{aligned}\tilde{C}_{k,\gamma} &= 4 \sin(2\gamma) \\ \tilde{c}_{k,\gamma} &= 2 \sin(2(\gamma - \pi/k)) + 2 \sin(2\gamma) = 4 \sin(2\gamma - \pi/k) \cos(\pi/k) \\ C_{k,\gamma} &= \tilde{c}_{k,\gamma} \\ c_{k,\gamma} &= 4 \sin(2(\gamma - \pi/k)).\end{aligned}$$

Then, for every $r_2 > 0$ and for $r_1 = \sigma r_2$, there are rectifiable 2-varifolds $V_1 = V_1^{r_1, r_2, k, \gamma}$, $V_2 = V_2^{r_1, r_2, k, \gamma}$ in \mathbb{R}^4 such that $\text{spt } \mu_{V_i} \subset A_{r_1}^{r_2}$,

$$\text{spt } V_i \subset G_2(A_{r_1}^{r_2}) \cap G_{\text{rad} \& J_{13}^{24}}^\varepsilon, \quad (54)$$

$$\begin{aligned}4 \sin 2(\gamma - \pi/k) \cdot \pi((s_2)^2 - (s_1)^2) &\leq \mathbf{M}(V_i \llcorner G_2(A_{s_1}^{s_2})) = V_i(G_2(A_{s_1}^{s_2})) \\ &\leq \frac{4}{\sin 2(\gamma - \pi/k)} \pi((s_2)^2 - (s_1)^2)\end{aligned} \quad (55)$$

whenever $r_1 \leq s_1 < s_2 \leq r_2$, and

$$\delta V_1(X) = \tilde{C}_{k,\gamma} B_{r_2, 2k}(X) - \tilde{c}_{k,\gamma} B_{r_1, k}(X) \quad (56)$$

$$\delta V_2(X) = C_{k,\gamma} B_{r_2, k}(X) - c_{k,\gamma} B_{r_1, 2k}(X) \quad (57)$$

where (denoting $N(x) = x/\|x\|$)

$$B_{\rho, k}(X) = \frac{1}{k} \sum_{i=1}^k \int_{S(\rho, 2i\pi/k)} X \cdot N \, d\mathcal{H}^2. \quad (58)$$

Proof Let $d > 0$ be such that

$$\begin{aligned}r_2 &= \sqrt{d / \cos 2\gamma} \\ r_1 &= \sigma r_2 = \sqrt{d / \cos 2(\gamma - \pi/k)}.\end{aligned}$$

Let $V_{t_1, t_2}^{d, \alpha_0}$ be as in Lemma 4.1, cf. (30) ($\alpha_0 \in \mathbb{R}$ and $\alpha_0 - \pi/4 < t_1 < t_2 < \alpha_0 + \pi/4$).

Let

$$V_{01} = \sum_{i=1}^k \left(V_{2i\pi/k, (2i+1)\pi/k}^{d, (2i+1)\pi/k - \gamma} + V_{(2i+1)\pi/k, (2i+2)\pi/k}^{d, (2i+1)\pi/k + \gamma} \right), \quad (59)$$

$$V_{02} = \sum_{i=1}^k \left(V_{(2i-1)\pi/k, 2i\pi/k}^{d, 2i\pi/k - \gamma} + V_{2i\pi/k, (2i+1)\pi/k}^{d, 2i\pi/k + \gamma} \right). \quad (60)$$

Then, from (31), (32) and (33),

$$\delta V_{01}(X) = 2 \sin(2\gamma) \sum_{i=1}^k \int_{S(r_2, (2i+1)\pi/k)} X \cdot N \, d\mathcal{H}^2 \quad (61)$$

$$- 2 \sin(2(\gamma - \pi/k)) \sum_{i=1}^k \int_{S(r_1, (2i+2)\pi/k)} X \cdot N \, d\mathcal{H}^2,$$

$$\delta V_{02}(X) = 2 \sin(2\gamma) \sum_{i=1}^k \int_{S(r_2, 2i\pi/k)} X \cdot N \, d\mathcal{H}^2 \quad (62)$$

$$- 2 \sin(2(\gamma - \pi/k)) \sum_{i=1}^k \int_{S(r_1, (2i+1)\pi/k)} X \cdot N \, d\mathcal{H}^2.$$

Let

$$V_{00} = V_{00}^{r_1, r_2, k} = \frac{1}{k} \sum_{i=1}^k V_{1, r_1, r_2, (\cos 2i\pi/k, \sin 2i\pi/k)} \quad (63)$$

where $V_{1, r_1, r_2, (a, b)} = V_{\mathcal{H}^2 \llcorner (\text{span}\{ae_1 + be_2, ae_3 + be_4\} \cap A_{r_1}^{r_2}(\mathbb{R}^4))}$ (see also (14), (15)). Since $\text{span}\{ae_1 + be_2, ae_3 + be_4\}$ is a linear space invariant under multiplication by J_{13}^{24} (see (22)), we have

$$\text{spt } V_{00} \subset G_2(A_{r_1}^{r_2}) \cap G_{\text{rad} \& J_{13}^{24}}^0. \quad (64)$$

Furthermore (cf. (5) or Section 3),

$$\begin{aligned} \delta V_{00}(X) &= \frac{1}{k} \sum_{i=1}^k \left(\int_{S(r_2, 2i\pi/k)} X \cdot N \, d\mathcal{H}^2 - \int_{S(r_1, 2i\pi/k)} X \cdot N \, d\mathcal{H}^2 \right) \\ &= B_{r_2, k}(X) - B_{r_1, k}(X). \end{aligned} \quad (65)$$

Let

$$V_1 = \frac{1}{k} V_{01} + 2 \sin(2\gamma) V_{00} \quad (66)$$

$$V_2 = \frac{1}{k} V_{02} + 2 \sin(2(\gamma - \pi/k)) V_{00} \quad (67)$$

Then the first variation of V_1 and V_2 is exactly as stated in (56), (57). Note that

$$\text{spt}_{V_{01}} \cup \text{spt}_{V_{02}} \subset G_{\text{rad} \& J_{13}^{24}}^\varepsilon$$

by Lemma 4.1, 5., and the same is true for planar varifold V_{00} , so also for V_1 and V_2 .

Let $r_1 \leq s_1 < s_2 \leq r_2$. We claim that

$$\mathbf{M}(V_{01} \llcorner G_2(A_{s_1}^{s_2})) = \mathbf{M}(V_{02} \llcorner G_2(A_{s_1}^{s_2})) = c 2k\pi((s_2)^2 - (s_1)^2) \quad (68)$$

where

$$\frac{1}{\sin 2\gamma} \leq c \leq \frac{1}{\sin 2(\gamma - \pi/k)}. \quad (69)$$

Indeed, if $\rho = \rho(d, s_1, s_2) \in [s_1, s_1] \subset [r_1, r_2]$ is as in Lemma 4.1, 6., then (68) holds true with

$$c = \frac{1}{\sqrt{1 - \frac{d^2}{\rho^4}}} \leq \frac{1}{\sqrt{1 - \frac{d^2}{(r_1)^4}}} = \frac{1}{\sqrt{1 - \cos^2 2(\gamma - \pi/k)}} = \frac{1}{\sin 2(\gamma - \pi/k)}.$$

On the other hand,

$$c \geq \frac{1}{\sqrt{1 - \frac{d^2}{(r_2)^4}}} = \frac{1}{\sqrt{1 - \cos^2 2\gamma}} = \frac{1}{\sin 2\gamma}.$$

We have exactly $\mathbf{M}(V_{00} \llcorner G_2(A_{s_1}^{s_2})) = \pi((s_2)^2 - (s_1)^2)$. Combining that with (69), we get (for $i = 1, 2$)

$$\begin{aligned} 4 \sin(2(\gamma - \pi/k)) \cdot \pi((s_2)^2 - (s_1)^2) &\leq \left(\frac{2}{\sin 2\gamma} + 2 \sin(2(\gamma - \pi/k)) \right) \pi((s_2)^2 - (s_1)^2) \\ &\leq \mathbf{M}(V_i \llcorner G_2(A_{s_1}^{s_2})) \\ &\leq \left(\frac{2}{\sin 2(\gamma - \pi/k)} + 2 \sin(2\gamma) \right) \pi((s_2)^2 - (s_1)^2) \\ &\leq \frac{4}{\sin 2(\gamma - \pi/k)} \pi((s_2)^2 - (s_1)^2). \end{aligned}$$

which is (55). \square

4.3 Layers.

Recall now that F is defined by (9) (see also (10) and (11)).

Lemma 4.3 *If $0 < R_1 < R_2 < R_3 < R_4 < \infty$ and $\varepsilon > 0$ then there is $c \in (1 - \varepsilon, 1)$ and a rectifiable 2-varifold V with $\text{spt } \mu_V \subset A_{R_1}^{R_4}$,*

$$\text{spt } V \subset G_2(A_{R_1}^{R_2} \cup A_{R_3}^{R_4}) \cap G_{\text{rad} \& J_{13}^{24}}^\varepsilon \quad (70)$$

$$\begin{aligned} &\cup G_2(A_{R_2}^{R_3}) \cap G_{\text{rad} \& J_{13}^{24}}^0 \\ &\subset G_2(A_{R_1}^{R_4}) \cap G_{\text{rad} \& J_{13}^{24}}^\varepsilon, \end{aligned} \quad (71)$$

$$(1 - \varepsilon) \pi((s_2)^2 - (s_1)^2) < \mathbf{M}(V \llcorner G_2(A_{s_1}^{s_2})) < (1 + \varepsilon) \pi((s_2)^2 - (s_1)^2) \quad (72)$$

whenever $R_1 \leq s_1 < s_2 \leq R_4$, and

$$\delta V(X) = B_{R_4, \infty}(X) - c B_{R_1, \infty}(X) \quad (73)$$

where (with $N(x) = x / \|x\|$)

$$B_{\rho, \infty}(X) = \int_{F((\rho \cdot S_1(\mathbb{R}^2)) \times S_1(\mathbb{R}^2))} X \cdot N \, d \frac{\mathcal{H}^2}{2\pi\rho}. \quad (74)$$

Proof Choose $k^{(n)} = 100 \cdot 2^n$ and $\gamma^{(n)} = \pi/4 - \pi/\sqrt{k^{(n)}}$.

With $C_{k,\gamma}$, $c_{k,\gamma}$, $\tilde{C}_{k,\gamma}$ and $\sigma_{k,\gamma}$ as in Lemma 4.2 we have

$$\begin{aligned} 1 &\geq \frac{C_{k^{(n)},\gamma^{(n)}}}{C_{k^{(n)},\gamma^{(n)}}} \geq \sin(2(\gamma^{(n)} - \pi/k^{(n)})) \geq 1 - 8\pi^2/k^{(n)} > 0, \\ 1 &\geq \frac{\tilde{C}_{k^{(n)},\gamma^{(n)}}}{\tilde{C}_{k^{(n)},\gamma^{(n)}}} \geq \sin(2\gamma^{(n)} - \pi/k^{(n)}) \cos(\pi/k^{(n)}) \geq 1 - 5\pi^2/k^{(n)} > 0. \end{aligned}$$

Hence

$$\prod_{n=1}^{\infty} \frac{C_{k^{(n)},\gamma^{(n)}}}{C_{k^{(n)},\gamma^{(n)}}} \in (0, 1), \quad \prod_{n=1}^{\infty} \frac{\tilde{C}_{k^{(n)},\gamma^{(n)}}}{\tilde{C}_{k^{(n)},\gamma^{(n)}}} \in (0, 1).$$

Furthermore

$$\begin{aligned} 0 &\leq 1 - \sigma_{k^{(n)},\gamma^{(n)}}^2 = 1 - \frac{\sin 2\pi/\sqrt{k^{(n)}}}{\sin(2\pi/k^{(n)} + 2\pi/\sqrt{k^{(n)}})} \\ &= \frac{2\cos(\pi/k^{(n)} + 2\pi/\sqrt{k^{(n)}}) \sin \pi/k^{(n)}}{\sin(2\pi/k^{(n)} + 2\pi/\sqrt{k^{(n)}})} \leq \pi \frac{\pi/k^{(n)}}{2\pi/k^{(n)} + 2\pi/\sqrt{k^{(n)}}} \leq \frac{\pi}{2} \frac{1}{\sqrt{k^{(n)}}}, \end{aligned}$$

hence

$$\left(\prod_{n=1}^{\infty} \sigma_{k^{(n)},\gamma^{(n)}} \right)^2 = \prod_{n=1}^{\infty} \sigma_{k^{(n)},\gamma^{(n)}}^2 \in (0, 1).$$

Choose $n_0 \in \mathbb{N}$ so that (for $n \geq n_0$)

$$\begin{aligned} \varepsilon_n &:= 2\cos 2(\gamma^{(n)} - \pi/k^{(n)}) < \varepsilon, \\ \sin 2(\gamma^{(n)} - \pi/k^{(n)}) &> 1 - \varepsilon/3, \end{aligned} \tag{75}$$

$$M = \frac{1}{4\sin(2\gamma^{(n_0)} - \pi/k^{(n_0)}) \cos(\pi/k^{(n_0)})} \frac{4}{\sin 2(\gamma^{(n_0)} - \pi/k^{(n_0)})} < 1 + \varepsilon, \tag{76}$$

$$c_1 := \prod_{n=n_0}^{\infty} \frac{\tilde{C}_{k^{(n)},\gamma^{(n)}}}{\tilde{C}_{k^{(n)},\gamma^{(n)}}} \in (1 - \varepsilon/3, 1),$$

$$c_2 := \prod_{n=n_0}^{\infty} \frac{C_{k^{(n)},\gamma^{(n)}}}{C_{k^{(n)},\gamma^{(n)}}} \in (1 - \varepsilon/3, 1)$$

and

$$\sigma := \prod_{n=n_0}^{\infty} \sigma_{k^{(n)},\gamma^{(n)}} \in (\max(R_1/R_2, R_3/R_4), 1).$$

Let $r^{(n_0)} := R_1/\sigma$, $R^{(n_0)} := \sigma R_4$, and then inductively $r^{(n+1)} := \sigma_{k^{(n)},\gamma^{(n)}} r^{(n)}$, $R^{(n+1)} := R^{(n)}/\sigma_{k^{(n)},\gamma^{(n)}}$. Then $\lim_{n \rightarrow \infty} r^{(n)} = R_1$, $\lim_{n \rightarrow \infty} R^{(n)} = R_4$,

$$\begin{aligned} R_1 &< r^{(n_0)} \leq R_2 \leq R_3 \leq R^{(n_0)} < R_4, \\ R_1 &< \dots < r^{(n_0+2)} < r^{(n_0+1)} < r^{(n_0)} \leq R^{(n_0)} < R^{(n_0+1)} < R^{(n_0+2)} < \dots < R_4. \end{aligned}$$

Let

$$c_{1,n} := \prod_{m=n}^{\infty} \frac{\tilde{C}_{k^{(m)}, \gamma^{(m)}}}{\tilde{C}_{k^{(m)}, \gamma^{(m)}}} \quad (\text{hence } c_1 = c_{1,n_0})$$

$$c_{2,n} := \prod_{m=n_0}^{n-1} \frac{C_{k^{(m)}, \gamma^{(m)}}}{C_{k^{(m)}, \gamma^{(m)}}} \quad (c_{2,n_0} := 1; c_2 = c_{2,\infty}),$$

and let $V_1^{r,s,k,\gamma}, V_2^{r,s,k,\gamma}$ be as in Lemma 4.2 and $V_{00} = V_{00}^{r^{(n_0)}, R^{(n_0)}, k^{(n_0)}}$ is as in (63). Let

$$V = \sum_{n=n_0}^{\infty} \frac{c_1 c_{2,n}}{C_{k^{(n)}, \gamma^{(n)}}} V_2^{r^{(n+1)}, r^{(n)}, k^{(n)}, \gamma^{(n)}} + c_1 V_{00} + \sum_{n=n_0}^{\infty} \frac{c_{1,n+1}}{\tilde{C}_{k^{(n)}, \gamma^{(n)}}} V_1^{R^{(n)}, R^{(n+1)}, k^{(n)}, \gamma^{(n)}}$$

and

$$V_m = \sum_{n=n_0}^m \frac{c_1 c_{2,n}}{C_{k^{(n)}, \gamma^{(n)}}} V_2^{r^{(n+1)}, r^{(n)}, k^{(n)}, \gamma^{(n)}} + c_1 V_{00} + \sum_{n=n_0}^m \frac{c_{1,n+1}}{\tilde{C}_{k^{(n)}, \gamma^{(n)}}} V_1^{R^{(n)}, R^{(n+1)}, k^{(n)}, \gamma^{(n)}}.$$

Then (70) can be obtained from (54) and (64).

Denote also

$$V - V_m := \sum_{n=m+1}^{\infty} \frac{c_1 c_{2,n}}{C_{k^{(n)}, \gamma^{(n)}}} V_2^{r^{(n+1)}, r^{(n)}, k^{(n)}, \gamma^{(n)}} + \sum_{n=m+1}^{\infty} \frac{c_{1,n+1}}{\tilde{C}_{k^{(n)}, \gamma^{(n)}}} V_1^{R^{(n)}, R^{(n+1)}, k^{(n)}, \gamma^{(n)}}.$$

Note that

$$\frac{c_1 c_{2,n}}{C_{k^{(n)}, \gamma^{(n)}}} \frac{4}{\sin 2(\gamma^{(n)} - \pi/k^{(n)})} \leq M, \quad n \geq n_0,$$

$$c_1 \leq 1 \leq M,$$

$$\frac{c_{1,n+1}}{\tilde{C}_{k^{(n)}, \gamma^{(n)}}} \frac{4}{\sin 2(\gamma^{(n)} - \pi/k^{(n)})} \leq M, \quad n \geq n_0.$$

Hence, by (55) and (76),

$$\begin{aligned} \mathbf{M}(V) &\leq \sum_{n=n_0}^{\infty} M\pi((r^{(n)})^2 - (r^{(n+1)})^2) \\ &\quad + M\pi((R^{(n_0)})^2 - (r^{(n_0)})^2) + \sum_{n=n_0}^{\infty} M\pi((R^{(n+1)})^2 - (R^{(n)})^2) \\ &= M\pi((R_4)^2 - (R_1)^2) \leq (1 + \varepsilon)\pi((R_4)^2 - (R_1)^2). \end{aligned} \tag{77}$$

In particular, V is a Radon measure. Therefore V is a varifold, obviously rectifiable. Moreover, $\mathbf{M}(V - V_n) \rightarrow 0$ as $n \rightarrow \infty$.

Note also that

$$\frac{4c_1 c_{2,n}}{C_{k^{(n)}, \gamma^{(n)}}} \geq c_1 c_2, \quad n \geq n_0,$$

$$c_1 \geq c_1 c_2,$$

$$\frac{4c_{1,n+1}}{\tilde{C}_{k^{(n)}, \gamma^{(n)}}} \geq c_1 c_2, \quad n \geq n_0.$$

Again by (55) (and (75)), we get

$$\begin{aligned} \mathbf{M}(V) &\geq \sum_{n=n_0}^{\infty} (1-\varepsilon/3)c_1c_2\pi((r^{(n)})^2 - (r^{(n+1)})^2) \\ &\quad + (1-\varepsilon/3)c_1c_2\pi((R^{(n_0)})^2 - (r^{(n_0)})^2) \\ &\quad + \sum_{n=n_0}^{\infty} (1-\varepsilon/3)c_1c_2\pi((R^{(n+1)})^2 - (R^{(n)})^2) \\ &= (1-\varepsilon/3)c_1c_2\pi((R_4)^2 - (R_1)^2) \geq (1-\varepsilon)\pi((R_4)^2 - (R_1)^2). \end{aligned} \quad (78)$$

From (77) and (78), (72) follows in the special case $s_1 = R_1$, $s_2 = R_4$. (Note that a special case $s_1 = r_1$, $s_2 = r_2$ of (55) was used.) Proof of the general case $R_1 \leq s_1 < s_2 \leq R_4$ of (72) is similar, with the following differences: a) some of the terms in (77), (78) might be replaced by 0, and b) some (at most two) of the terms might be “cut” to a smaller span between radii; the general case of (55) is used in such a case. For example, (78) is to be replaced by

$$\begin{aligned} \mathbf{M}(V \llcorner G_2(A_{s_1}^{s_2})) &\geq \sum_{n=n_0}^{\infty} (1-\varepsilon/3)c_1c_2\pi((\widehat{r^{(n)}})^2 - (\widehat{r^{(n+1)}})^2) \\ &\quad + (1-\varepsilon/3)c_1c_2\pi((\widehat{R^{(n_0)}})^2 - (\widehat{r^{(n_0)}})^2) \\ &\quad + \sum_{n=n_0}^{\infty} (1-\varepsilon/3)c_1c_2\pi((\widehat{R^{(n+1)}})^2 - (\widehat{R^{(n)}})^2) \\ &= (1-\varepsilon/3)c_1c_2\pi((s_2)^2 - (s_1)^2) \geq (1-\varepsilon)\pi((s_2)^2 - (s_1)^2). \end{aligned} \quad (79)$$

where $\widehat{\rho} = \min(\max(s_1, \rho), s_2)$.

We have $V_{n_0-1} = c_1V_{00}$ and, by (65),

$$\delta V_{n_0-1} = c_1 B_{R^{(n_0)}, k^{(n_0)}} - c_1 B_{r^{(n_0)}, k^{(n_0)}} = c_{1,n_0} B_{R^{(n_0)}, k^{(n_0)}} - c_{1,n_0} B_{r^{(n_0)}, k^{(n_0)}}$$

where $B_{\rho,k}$ is as in (58). Using (56) (57) we obtain by induction

$$\delta V_n = c_{1,n+1} B_{R^{(n+1)}, k^{(n+1)}} - c_{1,n+1} B_{r^{(n+1)}, k^{(n+1)}}. \quad (80)$$

Indeed, for $n \geq n_0$,

$$\begin{aligned} \delta V_n &= \frac{c_{1,n+1}}{\tilde{C}_{k^{(n)}, \gamma^{(n)}}} (\tilde{C}_{k^{(n)}, \gamma^{(n)}} B_{R^{(n+1)}, 2k^{(n)}} - \tilde{C}_{k^{(n)}, \gamma^{(n)}} B_{R^{(n)}, k^{(n)}}) + \\ &\quad c_{1,n} B_{R^{(n)}, k^{(n)}} - c_{1,n} B_{r^{(n)}, k^{(n)}} + \\ &\quad \frac{c_{1,n}}{C_{k^{(n)}, \gamma^{(n)}}} (C_{k^{(n)}, \gamma^{(n)}} B_{r^{(n)}, k^{(n)}} - c_{k^{(n)}, \gamma^{(n)}} B_{r^{(n+1)}, 2k^{(n)}}) \\ &= c_{1,n+1} B_{R^{(n+1)}, k^{(n+1)}} - c_{1,n+1} B_{r^{(n+1)}, k^{(n+1)}}. \end{aligned}$$

It is easy to verify that, for every smooth vector field X ,

$$B_{R^{(n+1)}, k^{(n+1)}}(X) \rightarrow B_{R_4, \infty}(X)$$

and

$$B_{r^{(n+1)}, k^{(n+1)}}(X) \rightarrow B_{R_1, \infty}(X).$$

On the other hand,

$$|\delta V(X) - \delta V_n(X)| \stackrel{(3)}{=} \left| \int \operatorname{div}_S X(x) d(V - V_n)(x, S) \right| \leq \|X\|_{C^1} \cdot \mathbf{M}(V - V_n) \rightarrow 0$$

as $n \rightarrow \infty$. From (80) we therefore obtain the formula for the first variation of V , with $c := \lim c_1 c_{2,n} = c_1 c_2 \in (1 - \varepsilon, 1)$. \square

Lemma 4.4 *If $0 < R_1 < R_2 < R_3 < R_4 < \infty$ and $\varepsilon > 0$ then there is $c \in (1 - \varepsilon, 1)$ and a rectifiable 2-varifold V with $\operatorname{spt} \mu_V \subset A_{R_1}^{R_4}$,*

$$\operatorname{spt} V \subset G_2(A_{R_1}^{R_2} \cup A_{R_3}^{R_4}) \cap G_{\operatorname{rad} \& J_{12}^{34}}^\varepsilon \quad (81)$$

$$\begin{aligned} & \cup G_2(A_{R_2}^{R_3}) \cap G_{\operatorname{rad} \& J_{12}^{34}}^0 \\ & \subset G_2(A_{R_1}^{R_4}) \cap G_{\operatorname{rad} \& J_{12}^{34}}^\varepsilon, \end{aligned} \quad (82)$$

$$(1 - \varepsilon)\pi((s_2)^2 - (s_1)^2) < \mathbf{M}(V \llcorner G_2(A_{s_1}^{s_2})) < (1 + \varepsilon)\pi((s_2)^2 - (s_1)^2) \quad (83)$$

whenever $R_1 \leq s_1 < s_2 \leq R_4$, and

$$\delta V(X) = B_{R_4, \infty}(X) - c B_{R_1, \infty}(X) \quad (84)$$

where $B_{\rho, \infty}$ is as in (74).

Proof The statement is the same as in Lemma 4.3, with the exception of a change of coordinates in (81) — we show that it is enough to exchange coordinates x_2 and x_3 . Let $\phi(x_1, x_2, x_3, x_4) = \phi(x_1, x_3, x_2, x_4)$, $((x_1, x_2, x_3, x_4) \in \mathbb{R}^4)$, and $\Phi(x, S) = (\phi(x), \phi(S))$ $((x, S) \in G_2(\mathbb{R}^4))$. Then $\phi(J_{13}^{24}x) = J_{12}^{34}\phi(x)$ and $\Phi(G_{\operatorname{rad} \& J_{13}^{24}}^\varepsilon) = G_{\operatorname{rad} \& J_{12}^{34}}^\varepsilon$ (cf. (24)). The domain of integration in (74) (which is parameterized by F) does not change under ϕ : $\phi(F((\rho a, \rho b), (c, d))) \stackrel{(11)}{=} F((c, d), (\rho a, \rho b)) \stackrel{(10)}{=} F((\rho c, \rho d), (a, b))$. Therefore, if \tilde{V} is as in Lemma 4.3, then $V := \phi_{\#} \tilde{V} = \Phi_{\#} \tilde{V}$ is a varifold with required properties. \square

Lemma 4.5 *If V is as in Lemma 4.3 or Lemma 4.4 and $r > 0$ then*

$$\mathbf{M}(V \llcorner G_2(S_r(\mathbb{R}^4))) = 0. \quad (85)$$

Proof For every $0 < \varepsilon_1 < r$ we have by (72), (83),

$$\mathbf{M}(V \llcorner G_2(A_{r-\varepsilon_1}^{r+\varepsilon_1})) \leq (1 + \varepsilon)\pi((r + \varepsilon_1)^2 - (r - \varepsilon_1)^2) \rightarrow 0.$$

\square

We do the last step of our construction of a stationary rectifiable varifold in the next section.

5 Two variants of the main result

Theorem 5.1 *There is a stationary rectifiable 2-varifold V in \mathbb{R}^4 that has a non-conical (hence non-unique) tangent at 0 and $0 < \theta^2(V, 0) < \infty$.*

Proof 1. The varifold V . For $0 < R_1 < R_2 < R_3 < R_4 < \infty$ and $\varepsilon > 0$ let

$$V_{R_1, R_2, R_3, R_4, \varepsilon}^1 \quad \text{and} \quad c_{R_1, R_2, R_3, R_4, \varepsilon}^1 \in (1 - 1/\varepsilon, 1)$$

denote the varifold and the number from Lemma 4.3. Let

$$V_{R_1, R_2, R_3, R_4, \varepsilon}^2 \quad \text{and} \quad c_{R_1, R_2, R_3, R_4, \varepsilon}^2 \in (1 - 1/\varepsilon, 1)$$

denote the varifold and the number from Lemma 4.4.

For $n \in \mathbb{Z}$, let

$$\begin{aligned} \varepsilon^{(n)} &= 1/4(n^2 + 1) \\ R_1^{(n)} &= 2^{-n} \\ R_2^{(n)} &= (1 + \varepsilon^{(n)})2^{-n} \\ R_3^{(n)} &= (1 - \varepsilon^{(n)})2^{-n+1} \\ R_4^{(n)} &= 2^{-n+1} = R_1^{(n-1)}. \end{aligned}$$

Let

$$V^{(n)} = \begin{cases} V_{R_1^{(n)}, R_2^{(n)}, R_3^{(n)}, R_4^{(n)}, \varepsilon^{(n)}}^1 & \text{for } n \text{ even, and} \\ V_{R_1^{(n)}, R_2^{(n)}, R_3^{(n)}, R_4^{(n)}, \varepsilon^{(n)}}^2 & \text{for } n \text{ odd.} \end{cases}$$

Accordingly, let

$$c^{(n)} = \begin{cases} c_{R_1^{(n)}, R_2^{(n)}, R_3^{(n)}, R_4^{(n)}, \varepsilon^{(n)}}^1 & \text{for } n \text{ even, and} \\ c_{R_1^{(n)}, R_2^{(n)}, R_3^{(n)}, R_4^{(n)}, \varepsilon^{(n)}}^2 & \text{for } n \text{ odd.} \end{cases}$$

Let $C^{(0)} = 1$ and

$$C^{(n)} = \begin{cases} \prod_{k=0}^{n-1} c^{(k)} & \text{for } n > 0, \text{ and} \\ \prod_{k=n}^{-1} \frac{1}{c^{(k)}} & \text{for } n < 0. \end{cases}$$

Since $c^{(k)} \geq 1 - \varepsilon^{(k)}$ and $\sum_{k \geq 0} \varepsilon^{(k)} < \infty$, we have

$$C^{(\infty)} := \lim_{n \rightarrow \infty} C^{(n)} \in (0, \infty).$$

Define

$$V := \sum_{n \in \mathbb{Z}} C^{(n)} V^{(n)}.$$

By (72), (83),

$$\frac{\pi}{2}((R_4^{(n)})^2 - (R_1^{(n)})^2) \leq \mathbf{M}(V^{(n)}) \leq M^{(n)} := 2\pi((R_4^{(n)})^2 - (R_1^{(n)})^2). \quad (86)$$

Since $C^{(n)}$ is decreasing,

$$\sum_{n \geq -k} C^{(n)} \mathbf{M}(V^{(n)}) \leq \sum_{n \geq -k} C^{(-k)} M^{(n)} = C^{(-k)} 2\pi(R_4^{(-k)})^2 < \infty. \quad (87)$$

V is a Radon measure because, for every k ,

$$V(G_2(\{x : \|x\| < 2^k\})) \leq \sum_{n \geq -k} C^{(n)} \mathbf{M}(V^{(n)}) < \infty.$$

Obviously, the varifold V is rectifiable.

Using (72) and (83) more wisely than in (86) we get that

$$C^{(\infty)}(1 - \varepsilon^{(n)})\pi R^2 \leq V(G_2(\{x : \|x\| \leq R\})) \leq C^{(n)}(1 + \varepsilon^{(n)})\pi R^2 \quad (88)$$

whenever $R \in (0, R_4^{(n)})$. Hence

$$\theta^2(V, 0) = C^{(\infty)}\pi \in (0, \infty).$$

2. The varifold V is stationary. Let X be a compactly supported smooth vector field on \mathbb{R}^4 . Fix $k \in \mathbb{N}$ such that $\text{spt} X \subset \{x : \|x\| < 2^k\}$. We have

$$|\delta V^{(n)}(X)| = \left| \int \text{div}_S X(x) dV^{(n)}(x, S) \right| \leq \|X\|_{C^1} \cdot \mathbf{M}(V^{(n)}) \leq \|X\|_{C^1} C^{(n)} M^{(n)}.$$

Since $\sum_{n \geq -k} C^{(n)} M^{(n)}$ converges by (87), we have

$$\begin{aligned} \delta V(X) &= \int \text{div}_S X(x) dV(x, S) \\ &= \sum_{n \geq -k} C^{(n)} \int \text{div}_S X(x) dV^{(n)}(x, S) = \sum_{n \geq -k} C^{(n)} \delta V^{(n)}(X). \end{aligned} \quad (89)$$

When using (73) and (84) to calculate $\sum_{n=-k}^m C^{(n)} \delta V^{(n)}(X)$ we realize that the first term is zero since integrating outside the support of X , next terms mutually cancel ($C^{(n)} c^{(n)} = C^{(n+1)}$, $R_1^{(n)} = R_4^{(n+1)}$) and the last one can be transformed so that we see it converges to 0. Formally,

$$\begin{aligned} \sum_{n=-k}^m C^{(n)} \delta V^{(n)}(X) &= \sum_{n=-k}^m \left(C^{(n)} B_{R_4^{(n)}, \infty}(X) - C^{(n)} c^{(n)} B_{R_1^{(n)}, \infty}(X) \right) \\ &= C^{(-k)} B_{R_4^{(-k)}, \infty}(X) - C^{(m)} c^{(m)} B_{R_1^{(m)}, \infty}(X) \\ &\stackrel{(74)}{=} -C^{(m)} c^{(m)} \int_{F((R_1^{(m)}, S_1(\mathbb{R}^2)) \times S_1(\mathbb{R}^2))} X \cdot N \frac{d\mathcal{H}^2}{2\pi R_1^{(m)}} \\ &\stackrel{x=R_1^{(m)}u}{=} -C^{(m+1)} \int_{F(S_1(\mathbb{R}^2) \times S_1(\mathbb{R}^2))} X(R_1^{(m)}u) \cdot N(u) \frac{d\mathcal{H}^2(u)}{2\pi} \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$ since $\lim R_1^{(m)} = 0$, $\lim C^{(m+1)} = C^{(\infty)}$, and mainly $X(\rho u) \rightarrow X(0)$ uniformly as $\rho \rightarrow 0$ and

$$\int_{F(S_1(\mathbb{R}^2) \times S_1(\mathbb{R}^2))} N(u) \frac{d\mathcal{H}^2(u)}{2\pi} = 0.$$

Therefore the sum in (89) is zero, $\delta V(X) = 0$ for arbitrary smooth compactly supported X , and V is a stationary varifold.

3. The tangents to V . First we describe (without proof) the tangents to V :

$$\text{VarTan}_0 V = \underbrace{\{(C^{(\infty)}/2\pi) V_{\{\zeta R_1^{(-i)}\}_{i \in \mathbb{Z}}}\}}_{V_\zeta} : \zeta > 0\}$$

where $R_1^{(i)}$ is as above and $V_{\{r_i\}}$ as in (19), (12), (13). Due to a “periodicity”, ζ can be restricted to $[R_1^{(0)}, R_1^{(-2)}) = [1, 4)$. Then V_ζ are mutually different and therefore not conical (cf. Lemma 3.1).

For the proof of the theorem we do not need anything more than to pick out a single tangent varifold and show that it is not conical. Let $\lambda_i = 4^{-i}$. Then $\lambda_i R_1^{(n)} = R_1^{(n+2i)}$ and (see (71), (82))

$$\text{spt} \left(\eta_{0, \lambda_i} \# V^{(n+2i)} \right) \subset G_2(A_{R_1^{(n)}}^{R_4^{(n)}}) \cap G_{\text{rad} \& D_{n+2i}}^{\varepsilon^{(n+2i)}}$$

where D_n is either symbol J_{13}^{24} (n even) or J_{12}^{34} (n odd). Therefore $D_{n+2i} = D_n$ and

$$\text{spt} \left(\eta_{0, \lambda_i} \# V \right) \subset \bigcup_{n \in \mathbb{Z}} \left(G_2(A_{R_1^{(n)}}^{R_4^{(n)}}) \cap G_{\text{rad} \& D_n}^{\varepsilon^{(n+2i)}} \right).$$

From (86),

$$\begin{aligned} \mathbf{M} \left((\eta_{0, \lambda_i} \# V) \llcorner G_2(A_{R_1^{(0)}}^{R_4^{(0)}}) \right) &= (\lambda_i)^{-2} \mathbf{M} \left(V \llcorner G_2(A_{R_1^{(2i)}}^{R_4^{(2i)}}) \right) \\ &= (\lambda_i)^{-2} C^{(2i)} \mathbf{M}(V^{(2i)}) \geq (\lambda_i)^{-2} C^{(\infty)} \frac{\pi}{2} ((R_4^{(2i)})^2 - (R_1^{(2i)})^2) = \frac{3\pi}{2} C^{(\infty)}. \end{aligned} \quad (90)$$

By the compactness theorem for Radon measures ([6, p. 242, p. 22]), there is a varifold C and a subsequence of $\{\lambda_i\}$ (denoted by $\{\lambda_i\}$ again) such that $\eta_{0, \lambda_i} \# V \rightarrow C$. (We note without proof that in fact it is not necessary to pass to a subsequence since even the original sequence is convergent.) Hence $C \in \text{VarTan}_0 V$. From the above,

$$\mathbf{M} \left(C \llcorner G_2(A_{R_1^{(0)}}^{R_4^{(0)}}) \right) \geq \frac{3\pi}{2} C^{(\infty)} > 0 \quad (91)$$

and

$$\text{spt} C \subset \bigcup_{n \in \mathbb{Z}} \left(G_2(A_{R_1^{(n)}}^{R_4^{(n)}}) \cap G_{\text{rad} \& D_n}^{\varepsilon} \right)$$

for every $\varepsilon > 0$ and thus also for $\varepsilon = 0$. In particular

$$\text{spt} C \cap G_2(\text{int} A_{R_1^{(0)}}^{R_4^{(0)}}) \subset G_{\text{rad} \& D_0}^0, \quad (92)$$

$$\text{spt} C \cap G_2(\text{int} A_{R_1^{(1)}}^{R_4^{(1)}}) \subset G_{\text{rad} \& D_1}^0 \quad (93)$$

where $\text{int} M$ denotes the interior of M . From (93),

$$\text{spt}(\eta_{0,1/2} \# C) \cap G_2(\text{int} A_{R_1^{(0)}}^{R_4^{(0)}}) \subset G_{\text{rad} \& D_1}^0. \quad (94)$$

Assume that C is conical. Then $\text{spt} C = \text{spt}(\eta_{0,1/2} \# C)$. Since $G_{\text{rad} \& D_0}^0$ and $G_{\text{rad} \& D_1}^0$ are disjoint (see (26)), we see that (92) and (94) is possible only when

$$\text{spt} C \cap G_2(\text{int} A_{R_1^{(0)}}^{R_4^{(0)}}) = \emptyset$$

which is a contradiction with (91) and (85). Hence C is not conical. \square

Theorem 5.2 *There is a stationary rectifiable 2-varifold V in \mathbb{R}^4 that has at least two different conical tangents at 0 and $0 < \theta^2(V, 0) < \infty$.*

Proof For $n \in \mathbb{Z}$, let

$$\begin{aligned} \varepsilon^{(n)} &= 1/4(n^2 + 1) \\ R_1^{(n)} &= 2^{-n^3} \\ R_2^{(n)} &= (1 + \varepsilon^{(n)})R_1^{(n)} \\ R_3^{(n)} &= (1 - \varepsilon^{(n)})R_4^{(n)} \\ R_4^{(n)} &= R_1^{(n-1)}. \end{aligned}$$

Note that $\{n^3\}$ is a strictly increasing sequence with increments at least one, hence $R_1^{(n)} < R_2^{(n)} < R_3^{(n)} < R_4^{(n)}$. Repeating the construction of Theorem 5.1 we obtain a rectifiable stationary 2-varifold V , but now the varifold's tangents at 0 are different.

Without proof we claim that, with $c = C^{(\infty)}/2\pi$, $cV_{1,0,\infty}$ and $cV_{2,0,\infty}$ (see Section 3, (12), (13)) are two different (Lemma 3.1) conical tangent varifolds to V at $0 \in \mathbb{R}^4$. There are also tangent varifolds of the form $c(V_{1,0,\rho} + V_{2,\rho,\infty})$ and $c(V_{2,0,\rho} + V_{1,\rho,\infty})$, $\rho > 0$; they are not conical, but they are “conical near 0”.⁵

We will give the detailed proof for existence of two different conical tangent varifolds at 0. Let $\lambda_i = iR_1^{(2i)}$ and $\tilde{\lambda}_i = iR_1^{(2i+1)}$.

⁵ We believe a slightly more complicated construction gives an example of a varifold whose all tangents are conical but the tangent at a point is non-unique. Basically, $\{J_{13}^{24}, J_{12}^{34}\}$ has to be replaced by a curve $\{J(t) : t \in [0, 1]\}$. A varifold would be used that takes directions in $G_{\text{rad} \& J(j/2^k)}^{1/(n^2+1)}$ on $A_{R_1^{(n)}}^{R_4^{(n)}}(\mathbb{R}^4)$ whenever $|n| = 2^k + j > 2$, $k, j \in \mathbb{N}$, $j \leq 2^k$.

Note that, for $i \rightarrow \infty$, $R_1^{(2i)}/\lambda_i = 1/i \rightarrow 0$ while $R_4^{(2i)}/\lambda_i = 2^{-(2i-1)^3+(2i)^3}/i \rightarrow \infty$. We have

$$\text{spt}(\eta_{0,\lambda_i} \# V^{(2i)}) \subset G_2(A_{R_1^{(2i)}/\lambda_i}^{R_4^{(2i)}/\lambda_i}) \cap G_{\text{rad} \& D_{2i}}^{\varepsilon^{(2i)}}$$

where $D_{2i} = D_0$ is the symbol “ J_{13}^{24} ”. Hence

$$\text{spt}(\eta_{0,\lambda_i} \# V) \subset G_2(A_0^{R_1^{(2i)}/\lambda_i}) \cup \left(G_2(A_{R_1^{(2i)}/\lambda_i}^{R_4^{(2i)}/\lambda_i}) \cap G_{\text{rad} \& D_0}^{\varepsilon^{(2i)}} \right) \cup G_2(A_{R_4^{(2i)}/\lambda_i}^\infty). \quad (95)$$

As in the proof of the previous theorem, we pass to a subsequence (denoted by $\{\lambda_i\}$ again) if necessary, so that $\eta_{0,\lambda_i} \# V \rightarrow C \in \text{VarTan}_0 V$ and $\eta_{0,\tilde{\lambda}_i} \# V \rightarrow \tilde{C} \in \text{VarTan}_0 V$.

By (95),

$$\text{spt} C \subset G_2(\{0\}) \cup \bigcap_{\varepsilon > 0} G_{\text{rad} \& D_0}^\varepsilon = G_2(\{0\}) \cup G_{\text{rad} \& D_0}^0.$$

By the same argument,

$$\text{spt} \tilde{C} \subset G_2(\{0\}) \cup G_{\text{rad} \& D_1}^0$$

where $D_1 = “J_{12}^{34}”$. Hence $C = \tilde{C}$ is possible (cf. again (26)) only if $\text{spt} C \cup \text{spt} \tilde{C} \subset G_2(\{0\})$. However, for sufficiently large $i \in \mathbb{N}$ we have $R_4^{(2i)}/\lambda_i > 2$, $R_1^{(2i)}/\lambda_i < 1$ and, by (72) and (83),

$$\begin{aligned} \mathbf{M}((\eta_{0,\lambda_i} \# V) \llcorner G_2(A_1^2)) &= (\lambda_i)^{-2} \mathbf{M}(V \llcorner G_2(A_{\lambda_i}^{2\lambda_i})) \\ &= (\lambda_i)^{-2} C^{(2i)} \mathbf{M}(V^{(2i)} \llcorner G_2(A_{\lambda_i}^{2\lambda_i})) \\ &\geq (\lambda_i)^{-2} C^{(\infty)} \frac{\pi}{2} ((2\lambda_i)^2 - (\lambda_i)^2) = \frac{3\pi}{2} C^{(\infty)} > 0 \end{aligned}$$

and therefore $C \neq \tilde{C}$ are two different conical tangents to V . \square

References

1. W. K. Allard, *On the first variation of a varifold*, Ann. of Math. (2) 95:3, 417–491 (1972)
2. S. Brendle, *Embedded minimal tori in S^3 and the Lawson conjecture*, preprint, <http://arxiv.org/abs/1203.6597>
3. H. Federer, *Geometric Measure Theory*, Springer 1969
4. T. C. O’Neil, *Geometric measure theory*, online at Encyclopedia of Mathematics, http://www.encyclopediaofmath.org/index.php?title=Geometric_measure_theory&oldid=28204 (First appeared in Supplement III of the Encyclopedia of Mathematics, Kluwer Academic Publishers, 2002)
5. R. Osserman, *Minimal varieties*, Bull. Amer. Math. Soc. Volume 75, Number 6 (1969), 1092–1120.
6. L. Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Vol. 3, Australian National University Canberra (1983). MR 0756417